Power System Analysis

Chapter 12 Optimal power flow

Steven Low EE/CS/EST 135 Caltech

Outline

1. Bus injection model
2. Branch flow model
3. OPF applications
4. Optimization algorithms

Outline

1. Bus injection model
    - Single-phase devices
    - Single-phase OPF
    - OPF as QCQP
    - Three-phase devices
    - Three-phase OPF
    - Three-phase OPF as QCQP
2. Branch flow model
3. OPF applications
4. Optimization algorithms

Single-phase devices

Voltage source j

- *Ideal* voltage source: terminal voltage V_j = internal voltage
- V_j is variable if the source is controllable, or given otherwise

Current source j

- *Ideal* current source: terminal voltage I_j = internal voltage
- I_j is variable if the source is controllable, or given otherwise

Power source j

- *Ideal* power source: terminal power s_j = internal power
- s_j is variable if the source is controllable, or given otherwise

Impedance j

- Impedance z_j: constrains its terminal voltage & current $V_j = -z_j I_j$

Single-phase OPF
Assumptions

Assume WLOG

- Single-phase devices: voltage sources and power sources only
- Each bus has a single device with $\left(V_j, s_j\right)$

Formulate the simplest OPF to study general computational properties

Single-phase OPF

Simplest formulation

Optimization variable: $(V, s) := \left(V_j, s_j, j \in \overline{N} \right)$

- Represents voltage sources V_j and power sources s_j only

Cost function $C_0(V, s)$

- Fuel cost : $C_0(V, s) := \sum_{j:\text{gens}} c_j \, \text{Re}(s_j)$
- Total real power loss: $C_0(V, s) := \sum_{j} \text{Re}(s_j)$

Single-phase OPF

Simplest formulation

Power flow equations in BIM

- Equality constraints on (V, s)

$$s_j \;=\; \sum_{k:j\sim k} S_{jk}(V) \;\;:=\;\; \sum_{k:j\sim k} \left(y_{jk}^s\right)^{\mathsf{H}} \left(|V_j|^2 - V_j V_k^{\mathsf{H}}\right) \;+\; \left(y_{jj}^m\right)^{\mathsf{H}} |V_j|^2, \qquad j \in \overline{N}$$

- Derivation:

$$I_{jk}(V) \;:=\; y_{jk}^s (V_j - V_k) \;+\; y_{jk}^m V_j$$

$$S_{jk}(V) \;:=\; V_j I_{jk}^{\mathsf{H}}(V) \;\;:=\;\; \left(y_{jk}^s\right)^{\mathsf{H}} \left(|V_j|^2 - V_j V_k^{\mathsf{H}}\right) \;+\; \left(y_{jk}^m\right)^{\mathsf{H}} |V_j|^2$$

- Can also use polar form and Cartesian form
- Nonlinear and global equality constraints, resulting in nonconvexity of OPF

Single-phase OPF

Simplest formulation

Operational constraints

- Injection limits (e.g. gen. or load capacity limits): $s_j^{\min} \ \leq \ s_j \ \leq \ s_j^{\max}$
- Voltage limits: $v_j^{\min} \ \leq \ |V_j|^2 \ \leq \ v_j^{\max}$
- Line limits: $|I_{jk}(V)|^2 \leq I_{jk}^{\max}, \ |I_{kj}(V)|^2 \leq I_{kj}^{\max}$

$$\left| y_{jk}^s (V_j - V_k) \ + \ y_{jk}^m V_j \right|^2 \ \leq \ I_{jk}^{\max}, \qquad (j,k) \in E$$

$$\left| y_{kj}^s (V_k - V_j) \ + \ y_{kj}^m V_k \right|^2 \ \leq \ I_{kj}^{\max}, \qquad (j,k) \in E$$

Line limits can also be on line powers $\left(S_{jk}(V), S_{kj}(V)\right)$ or apparent powers $\left(\left|S_{jk}(V)\right|, \left|S_{kj}(V)\right|\right)$

Single-phase OPF

Simplest formulation

OPF in BIM

$$\min_{(V,s)} \quad C_0(V,s)$$

$$\text{subject to} \quad f(V,s) = 0 \qquad \text{power flow equations}$$

$$g(V,s) \leq 0 \qquad \text{operational constraints}$$

- Does not need assumption $y_{jk}^s = y_{kj}^s$
- Can accommodate single-phase transformers with *complex* turns ratios

Single-phase OPF

1. Other devices
   - Can include other devices such as current sources, impedances, capacity taps
   - Allow multiple devices connected to same bus

2. Can formulate OPF in terms of V only
   - Use power flow equations to express injections $s_j(V)$ as functions of V
   - Eliminate s_j and power flow equations (equality constraints)

Next: explain each in turn

Single-phase OPF

Including other devices

Examples

- Current source (controllable): variable I_j with local constraints $|I_j|^2 \le I_j^{\max}, \quad s_j = V_j I_j^{\mathsf{H}}$
- Impedance z_j: imposes additional constraint $s_j = |V_j|^2 / z_j^{\mathsf{H}}$
- Capacitor tap (controllable): variable y_j with local constraints $y_j^{\min} \le y_j \le y_j^{\max}, \quad s_j = y_j^{\mathsf{H}} |V_j|^2$
- Multiple devices: injection variables s_{jk} with local constraints $s_{jk}^{\min} \le s_{jk} \le s_{jk}^{\max}, \quad s_j = \sum_k s_{jk}$

Including other devices at bus j imposes additional local constraints

- Additional optimization var u_j may be introduced
- Equality constraints relating $\left(V_j, s_j\right)$ and u_j (if present) : $f_j\left(V_j, s_j, u_j\right) = 0$
- Inequality (operational) constraints (e.g., capacity limits): $g_j(u_j) \le 0$

Single-phase OPF

In terms of V only

Equality constraints (BIM in complex form)

$$s_j(V) \;=\; \sum_{k:j\sim k} S_{jk}(V) \;:=\; \sum_{k:j\sim k} \left(y_{jk}^s\right)^{\mathsf{H}} \left(|V_j|^2 - V_j V_k^{\mathsf{H}}\right) + \left(y_{jj}^m\right)^{\mathsf{H}} |V_j|^2, \qquad j \in \overline{N}$$

- Expresses s_j in terms of voltages V

Cost $C_0(V) := C_0(V, s(V))$ expressed as function of V

- Fuel cost:

$$C_0(V) := \sum_{j:\text{gens}} c_j \,\text{Re}(s_j(V)) \;=\; \sum_{j:\text{gens}} c_j \,\text{Re}\left(\sum_{k:j\sim k} \left(y_{jk}^s\right)^{\mathsf{H}} \left(|V_j|^2 - V_j V_k^{\mathsf{H}}\right) + \left(y_{jj}^m\right)^{\mathsf{H}} |V_j|^2\right)$$

- Total real power loss:

$$C_0(V) := \sum_j \text{Re}(s_j(V))$$

Single-phase OPF
Operational constraints

Injection limits (e.g. generation or load capacity limits) $s_j^{\min} \leq \ s_j(V) \leq \ s_j^{\max}$:

$$\underline{s}_j \ \leq \ \sum_{k:j\sim k} \left(y_{jk}^s\right)^{\mathsf{H}} \ \left(|V_j|^2 - V_j V_k^{\mathsf{H}}\right) \ + \ \left(y_{jj}^m\right)^{\mathsf{H}} \ |V_j|^2 \ \leq \ \bar{s}_j, \qquad j \in \overline{N}$$

- Polar form:

$$\underline{p}_j \ \leq \ \left(\sum_{k=0}^{N} g_{jk}\right)|V_j|^2 \ - \ \sum_{k\neq j} |V_j|\,|V_k| \left(g_{jk}\cos\theta_{jk} - b_{jk}\sin\theta_{jk}\right) \ \leq \ \bar{p}_j$$

$$\underline{p}_j \ \leq \ \left(\sum_{k=0}^{N} b_{jk}\right)|V_j|^2 \ - \ \sum_{k\neq j} |V_j|\,|V_k| \left(b_{jk}\cos\theta_{jk} + g_{jk}\sin\theta_{jk}\right) \ \leq \ \bar{q}_j$$

Single-phase OPF

Operational constraints

Voltage limits (same as before):

$$v_j^{\min} \ \leq \ |V_j|^2 \ \leq \ v_j^{\max}, \qquad j \in \overline{N}$$

Line limits (same as before):

$$\left| y_{jk}^s (V_j - V_k) \ + \ y_{jk}^m V_j \right|^2 \leq \ I_{jk}^{\max}, \qquad (j,k) \in E$$

$$\left| y_{kj}^s (V_k - V_j) \ + \ y_{kj}^m V_k \right|^2 \leq \ I_{kj}^{\max}, \qquad (j,k) \in E$$

- Line limits can also be on line powers $\left(S_{jk}(V), S_{kj}(V) \right)$ or apparent powers $\left(\left| S_{jk}(V) \right|, \left| S_{kj}(V) \right| \right)$

Single-phase OPF

In terms of V only

Feasible set

$$\mathbb{V} := \left\{ V \in \mathbb{C}^{N+1} \mid V \text{ satisfies operational constraints} \right\}$$

OPF in BIM

$$\min_{V \in \mathbb{V}} \quad C_0(V)$$

- Does not need assumption $y_{jk}^s = y_{kj}^s$
- Can accommodate single-phase transformers with *complex* turns ratios

Single-phase OPF

In terms of V only

Feasible set

$$\mathbb{V} := \left\{ V \in \mathbb{C}^{N+1} \mid V \text{ satisfies operational constraints} \right\}$$

OPF in BIM

$$\min_{V \in \mathbb{V}} \quad C_0(V)$$

We will mostly study this simple OPF

Can express it as a QCQP

Outline

1. Bus injection model
   - Single-phase devices
   - Single-phase OPF
   - OPF as QCQP
   - Three-phase devices
   - Three-phase OPF
   - Three-phase OPF as QCQP
2. Branch flow model
3. Optimization algorithms

OPF as QCQP

QCQP

Quadratically constrained quadratic program:

$$\min_{x\in\mathbb{C}^n} \quad x^{\mathsf{H}} C_0 x$$

$$\text{s.t.} \quad x^{\mathsf{H}} C_l x \ \leq \ b_l, \qquad l = 1,\ldots,L$$

- C_l : $n \times n$ Hermitian matrix
- $b_l \in \mathbb{R}$
- Homogeneous QCQP : all monomials are of degree 2

OPF as QCQP

QCQP

Inhomogeneous QCQP

$$\min_{x \in \mathbb{C}^n} \quad x^{\mathsf{H}} C_0 x \; + \; \left(c_0^{\mathsf{H}} x + x^{\mathsf{H}} c_0 \right)$$

$$\text{s.t.} \quad x^{\mathsf{H}} C_l x \; + \; \left(c_l^{\mathsf{H}} x + x^{\mathsf{H}} c_l \right) \; \leq \; b_l, \qquad l = 1, \ldots, L$$

Homogenization:

- Idea: $|x|^2 + \left(c^{\mathsf{H}} x + x^{\mathsf{H}} c \right) \leq b \iff |x + ct|^2 - |c|^2 |t|^2 \leq b, \quad |t|^2 = 1$
- If $\left(x, t = e^{i\theta} \right)$ satisfies 2nd inequality, then $xt = xe^{i\theta}$ satisfies 1st inequality

OPF as QCQP

QCQP

Equivalent homogeneous QCQP

$$\min_{x\in\mathbb{C}^n,\, t\in\mathbb{C}} \quad \begin{bmatrix} x^{\mathsf H} & t^{\mathsf H} \end{bmatrix} \begin{bmatrix} C_0 & c_0 \\ c_0^{\mathsf H} & 0 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix}$$

$$\text{s.t.} \quad \begin{bmatrix} x^{\mathsf H} & t^{\mathsf H} \end{bmatrix} \begin{bmatrix} C_l & c_l \\ c_l^{\mathsf H} & 0 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \leq b_l, \qquad l = 1,\ldots,L$$

$$\begin{bmatrix} x^{\mathsf H} & t^{\mathsf H} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} = 1$$

Homogenization:

- Idea: $|x|^2 + \left(c^{\mathsf H}x + x^{\mathsf H}c\right) \leq b \quad \Longleftrightarrow \quad |x+ct|^2 - |c|^2|t|^2 \leq b, \quad |t|^2 = 1$
- If $\left(x, t = e^{i\theta}\right)$ satisfies 2nd inequality, then $xt = xe^{i\theta}$ satisfies 1st inequality

OPF as QCQP

To write OPF as QCQP:

- Assume cost function $C_0(V) = V^{\mathsf{H}} C_0 V$ can be written as a quadratic form
- Need to rewrite operational constraints in terms of quadratic forms

OPF as QCQP

Injection limits $s_j^{\min} \leq s_j(V) \leq s_j^{\max}$

$$s_j(V) = V_j I_j^{\mathsf{H}} = \left(e_j^{\mathsf{H}} V\right)\left(e_j^{\mathsf{H}} I\right)^{\mathsf{H}} = e_j^{\mathsf{H}} \, V V^{\mathsf{H}} Y^{\mathsf{H}} e_j$$

$$s_j(V) = \mathrm{tr}\left(e_j^{\mathsf{H}} V V^{\mathsf{H}} Y^{\mathsf{H}} e_j\right) = \mathrm{tr}\left(\left(Y^{\mathsf{H}} e_j e_j^{\mathsf{H}}\right) V V^{\mathsf{H}}\right) =: V^{\mathsf{H}} Y_j^{\mathsf{H}} V$$

OPF as QCQP

Injection limits $s_j^{\min} \leq s_j(V) \leq s_j^{\max}$

$$s_j(V) = V_j I_j^{\mathsf{H}} = \left(e_j^{\mathsf{H}} V\right)\left(e_j^{\mathsf{H}} I\right)^{\mathsf{H}} = e_j^{\mathsf{H}} \, V V^{\mathsf{H}} Y^{\mathsf{H}} e_j$$

$$s_j(V) = \operatorname{tr}\left(e_j^{\mathsf{H}} V V^{\mathsf{H}} Y^{\mathsf{H}} e_j\right) = \operatorname{tr}\left(\left(Y^{\mathsf{H}} e_j e_j^{\mathsf{H}}\right) V V^{\mathsf{H}}\right) =: V^{\mathsf{H}} Y_j^{\mathsf{H}} V$$

- Y_j is not Hermitian so $V^{\mathsf{H}} Y_j^{\mathsf{H}} V$ is generally complex
- Define $\Phi_j := \frac{1}{2}\left(Y_j^{\mathsf{H}} + Y_j\right), \qquad \Psi_j := \frac{1}{2i}\left(Y_j^{\mathsf{H}} - Y_j\right)$
- Then $\mathrm{Re}(s_j) = V^{\mathsf{H}} \Phi_j V, \qquad \mathrm{Im}(s_j) = V^{\mathsf{H}} \Psi_j V$

Hence $s_j^{\min} \leq s_j(V) \leq s_j^{\max}$ is equivalent to:

$$p_j^{\min} \leq V^{\mathsf{H}} \Phi_j V \leq p_j^{\max}, \qquad q_j^{\min} \leq V^{\mathsf{H}} \Psi_j V \leq q_j^{\max}$$

OPF as QCQP

Voltage limits

Voltage magnitude is: $|V_j|^2 = V^{\mathsf{H}} J_j V$ where $J_j := e_j e_j^{\mathsf{T}}$

Hence voltage limits are: $v_j^{\min} \leq V^{\mathsf{H}} J_j V \leq v_j^{\max}$

OPF as QCQP

Line limits

Write I_{jk} in terms of voltage vector V:

$$I_{jk} \;=\; y^s_{jk}(V_j - V_k) \;+\; y^m_{jk} V_j \;=\; \left(y^s_{jk}(e_j - e_k)^{\mathsf{T}} \;+\; y^m_{jk}\, e_j^{\mathsf{T}} \right) V$$

Hence current limit is: $\;|I_{jk}|^2 = V^{\mathsf{H}} \hat{Y}_{jk} V \;\leq\; I^{\max}_{jk}\;$ where

$$\hat{Y}_{jk} \;:=\; \left(y^s_{jk}(e_j - e_k)^{\mathsf{T}} \;+\; y^m_{jk}\, e_j^{\mathsf{T}} \right)^{\mathsf{H}} \left(y^s_{jk}(e_j - e_k)^{\mathsf{T}} \;+\; y^m_{jk}\, e_j^{\mathsf{T}} \right)$$

OPF as QCQP

Simplest formulation

$$
\begin{aligned}
\min_{V \in \mathbb{C}^{N+1}} \quad & V^{\mathsf{H}} C_0 V \\
\text{s.t.} \quad & p_j^{\min} \ \leq \ V^{\mathsf{H}} \Phi_j V \ \leq \ p_j^{\max}, && j \in \overline{N} \\
& q_j^{\min} \ \leq \ V^{\mathsf{H}} \Psi_j V \ \leq \ q_j^{\max}, && j \in \overline{N} \\
& v_j^{\min} \ \leq \ V^{\mathsf{H}} J_j V \ \leq \ v_j^{\max}, && j \in \overline{N} \\
& V^{\mathsf{H}} \hat{Y}_{jk} V \ \leq \ \bar{I}_{jk}^{\max}, && (j,k) \in E \\
& V^{\mathsf{H}} \hat{Y}_{kj} V \ \leq \ \bar{I}_{kj}^{\max}, && (j,k) \in E
\end{aligned}
$$

Outline

1. Bus injection model
    - Single-phase devices
    - Single-phase OPF
    - OPF as QCQP
    - Three-phase devices
    - Three-phase OPF
    - Three-phase OPF as QCQP
2. Branch flow model
3. Optimization algorithms

Recall: overall 3-phase BIM

Device + network

1. Device model for each 3-phase device
   - Internal model on $\left(V_j^{Y/\Delta}, I_j^{Y/\Delta}, s_j^{Y/\Delta}\right)$ + conversion rules
   - External model on $\left(V_j, I_j, s_j\right)$
   - Either can be used
   - Power source models are nonlinear; other devices are linear

Our perspective:

- Internal vars $\left(V_j^{Y/\Delta}, I_j^{Y/\Delta}, s_j^{Y/\Delta}\right)$ are controllable, depending on types of device
- External vars $\left(V_j, I_j, s_j\right)$ are not directly controllable

$\therefore$ use internal model + conversion rules

Recall: overall 3-phase BIM

Device + network

2. Network model relates terminal vars (V, I, s)

   - Nodal current balance (linear): $I = YV$
   - Nodal power balance (nonlinear):

$$s_j = \sum_{k:j\sim k} \text{diag}\left(V_j (V_j - V_k)^{\mathsf{H}} y_{jk}^{s\mathsf{H}} \; + \; V_j V_j^{\mathsf{H}} y_{jk}^{m\mathsf{H}} \right)$$

   - Either can be used

For OPF, our formulation uses (V, s):

- Relate (V, s) through power flow equations
- Power sources lead to nonlinear analysis, even if we use $I = YV$ as network equation
- Need to relate internal optimization vars to $\left(V_j, s_j\right)$ using conversion rules

Three-phase devices

Voltage source $V_j^{Y/\Delta}$

- *Internal* optimization variable $u_j := V_j^{Y/\Delta}$ (γ_j^Y assumed given)
- Local constraints that relate internal vars to $\left(V_j, s_j\right)$

$$Y: \qquad V_j = V_j^Y + \gamma_j^Y 1$$

$$\Delta: \qquad \Gamma V_j = V_j^\Delta$$

Note:

- Choosing V_j^Δ does not uniquely determine V_j
- Optimization over V_j implicitly chooses an optimal $\gamma_j^\Delta := \frac{1}{3} 1^\top V_j$
- If γ_j^Δ is given, then $\Gamma V_j = V_j^\Delta$ should be replaced by $V_j = \Gamma^\dagger V_j^\Delta + \gamma_j^\Delta 1$

Three-phase devices

Current source $I_j^{Y/\Delta}$

- *Internal* optimization variable $u_j := I_j^{Y/\Delta}$
- Local constraints that relate internal vars and $\left(V_j, s_j\right)$

$$Y: \qquad s_j \;=\; -\,\text{diag}\left(V_j I_j^{Y\mathsf{H}}\right)$$

$$\Delta: \qquad s_j \;=\; -\,\text{diag}\left(V_j I_j^{\Delta\mathsf{H}}\,\Gamma\right)$$

Note:

- Optimization over I_j^{Δ} implicitly chooses an optimal $\beta_j^{\Delta} := \frac{1}{3}1^{\mathsf{T}} I_j^{\Delta}$
- If β_j^{Δ} is given, it imposes an additional constraint $I_j^{\Delta} = -\frac{1}{3}\Gamma I_j + \beta_j^{\Delta}1$ (and express I_j in terms of $\left(V_j, s_j\right)$

Three-phase devices

Power source $\left(s_j^{Y/\Delta}, I_j^{Y/\Delta}\right)$

- *Internal* optimization variable $u_j := \left(s_j^{Y/\Delta}, I_j^{Y/\Delta}\right)$ (assume $\gamma_j^Y := V_j^n = 0$)
- Local constraints that relate internal vars and $\left(V_j, s_j\right)$

$$Y: \qquad s_j = -\, s_j^Y$$

$$\Delta: \qquad s_j = -\,\text{diag}\left(V_j I_j^{\Delta \mathsf{H}} \Gamma\right), \qquad s_j^{\Delta} = \text{diag}\left(\Gamma V_j I_j^{\Delta \mathsf{H}}\right)$$

Impedance $z_j^{Y/\Delta}$

- Given parameter: $z_j^{Y/\Delta}$ (assume $\gamma_j^Y := V_j^n = 0$)
- Local constraints on terminal vars $\left(V_j, s_j\right)$

$$Y: \qquad s_j = -\,\text{diag}\left(V_j V_j^{\mathsf{H}} y_j^{Y\mathsf{H}}\right)$$

$$\Delta: \qquad s_j = -\,\text{diag}\left(V_j V_j^{\mathsf{H}} Y_j^{\Delta \mathsf{H}}\right)$$

$$Y_j^{\Delta} := \Gamma^{\mathsf{T}} y^{\Delta} \Gamma$$

Three-phase OPF

Variables:

- Terminal variables $\left(V_j, s_j\right)$
- *Internal* variables u_j depending on devices (discussed above)

Cost function: $C_0\,(V, s, u)$

Equality constraints:

1. Power flow equations on (V, s) (global constraint): $f(V, s) = 0$

$$s_j \;=\; \sum_{k:j\sim k} \text{diag}\left(V_j(V_j - V_k)^H\left(y_{jk}^s\right)^H \;+\; V_jV_j^H\left(y_{jk}^m\right)^H\right), \qquad j \in \overline{N}$$

2. Conversion rules relating internal optimization var u_j to $\left(V_j, s_j\right)$ (local constraint, discussed above)

$$f_j^{Y/\Delta}\left(V_j, s_j, u_j\right) = 0, \qquad j \in \overline{N}$$

Three-phase OPF

Inequality constraints:

1. Operational constraints on external vars: $g(V, s) \leq 0$

injection limits:	$s_j^{\phi\,\min} \leq s_j^\phi \leq s_j^{\phi\,\max},$	$\phi \in \{a, b, c\}, \quad j \in \overline{N}$
voltage limits:	$v_j^{\phi\,\min} \leq \left\lvert V_j^\phi \right\rvert^2 \leq v_j^{\phi\,\max},$	$\phi \in \{a, b, c\}, \quad j \in \overline{N}$
line limits:	$\left\lvert I_{jk}^\phi(V) \right\rvert^2 \leq I_{jk}^{\phi\,\max}, \quad \left\lvert I_{kj}^\phi(V) \right\rvert^2 \leq I_{kj}^{\phi\,\max},$	$\phi \in \{a, b, c\}, \quad (j, k) \in E$

Same constraints as single-phase OPF, but on single-phase equivalent circuit

Three-phase OPF

Inequality constraints:

2. Operational constraints on internal vars: $g_j^{Y/\Delta}(u_j) \leq 0$

   for $\phi n \in \{an, bn, cn\}, \quad \phi\varphi \in \{ab, bc, ca\}$

   voltage source: $v_j^{\phi n \min} \leq \left|V_j^{\phi n}\right|^2 \leq v_j^{\phi n \max}, \qquad v_j^{\phi\varphi \min} \leq \left|V_j^{\phi\varphi}\right|^2 \leq v_j^{\phi\varphi \max}$

   current source: $\left|I_j^{\phi n}\right|^2 \leq I_j^{\max}, \qquad \left|I_j^{\phi\varphi}\right|^2 \leq I_j^{\max}$

   power source: $s_j^{\phi n \min} \leq s_j^{\phi n} \leq s_j^{\phi n \max}, \qquad \left|I_j^{\phi n}\right|^2 \leq I_j^{\phi n \max}$

   $s_j^{\phi\varphi \min} \leq s_j^{\phi\varphi} \leq s_j^{\phi\varphi \max}, \qquad \left|I_j^{\phi\varphi}\right|^2 \leq I_j^{\phi\varphi \max}$

   Local constraints at each bus j

Three-phase OPF

Constraints summary

1. Constraints on terminal variables: $f(V, s) = 0, \quad g(V, s) \leq 0$
   - Power flow equation and operational constraints (terminal power injection limits, voltage limits, line limits)
   - Global constraints
   - Extension of single-phase constraints to 3-phase setting, using single-phase equivalent
2. Conversion rules relating u_j and $\left(V_j, s_j\right)$: $f_j^{Y/\Delta}\left(u_j, V_j, s_j\right) = 0$
   - Local equality constraint for each device j
3. Operational constraints on internal variables: $g_j^{Y/\Delta}(u_j) \leq 0$
   - Depending on type of device (voltage and capacity limits)
   - Local constraints for each device j

Three-phase OPF

Simplest formulation

OPF in BIM

$$\min_{(V,s,u)} \quad C_0(V, s, u)$$

$$f(V, s) = 0, \qquad g(V, s) \leq 0$$

Global constraints on terminal vars

$$f_j^{Y/\Delta}(V_j, s_j, u_j) = 0, \qquad g_j^{Y/\Delta}(u_j) \leq 0, \qquad j \in \overline{N}$$

Local constraints at each bus j

Three-phase OPF
As QCQP

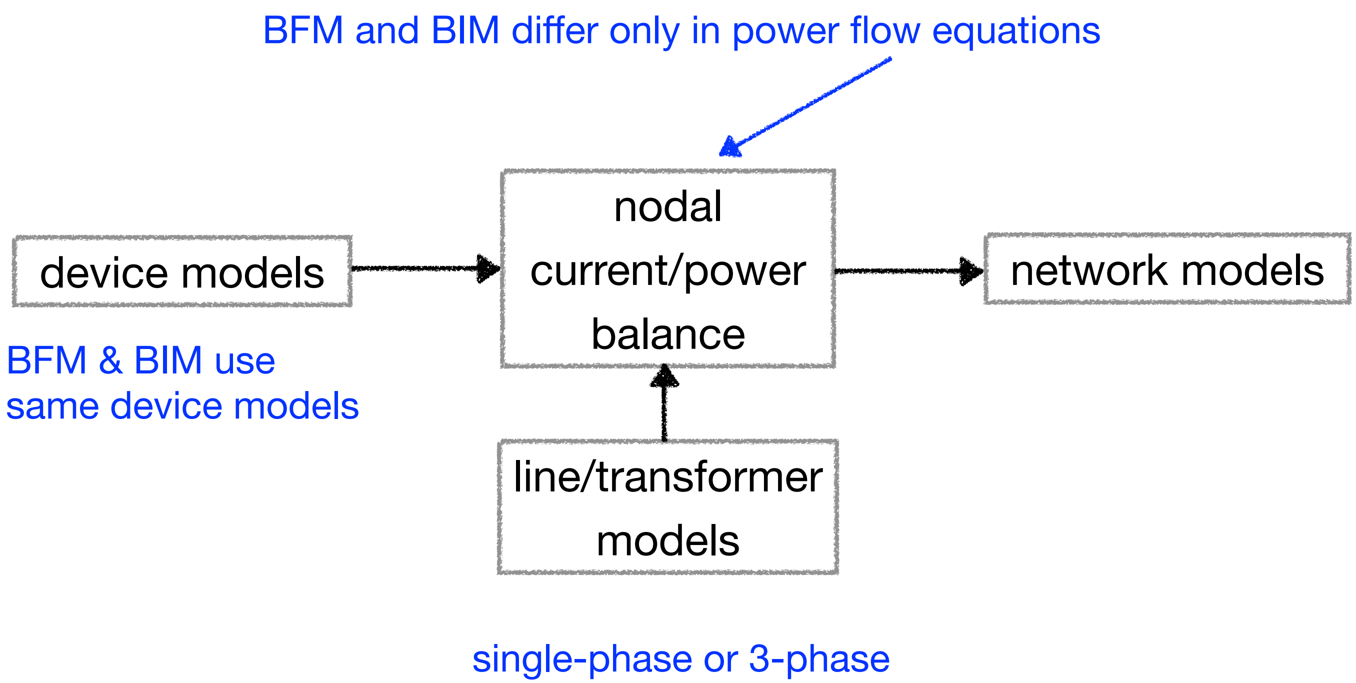
1. Can formulate OPF in terms of (V, u) only
   - Use power flow equations to express $s_j(V) = V^{\mathsf{H}} \left(Y_j^{\phi \mathsf{H}} \right) V$ and eliminate s_j and $f(V, s) = 0$
   - Same idea as before applied to single-phase equivalent
2. Can formulate OPF as QCQP
   - Express operational constraints $g(V, s(V)) \leq 0$ in terms of quadratic forms in V (same idea applied to single-phase equivalent)
   - Express conversion rules $f_j^{Y/\Delta} \left(V_j, s_j(V), u_j \right) = 0$ in terms of quadratic forms in (V, u_j)

For details: see Lecture Notes

Outline

1. Bus injection model
2. Branch flow model
   - Single-phase OPF
   - Three-phase OPF
3. OPF applications
4. Optimization algorithms

Overview


BFM and BIM differ only in power flow equations
device models
nodal current/power balance
network models
BFM & BIM use same device models
line/transformer models
single-phase or 3-phase

Assumptions

Both single-phase & 3-phase OPF

Radial network

- BFM most useful for modeling distribution systems

$z^s_{jk} = z^s_{kj}$ or equivalently $y^s_{jk} = y^s_{kj}$

- Does not include 3-phase transformers in ΔY or $Y\Delta$ configuration (or single-phase transformers with complex gains)

$y^m_{jk} = y^m_{kj} = 0$

- Reasonable assumption for distribution line where $|y^m_{jk}|, |y^m_{kj}| \ll |y^s_{jk}|$

Includes only voltage sources and power sources

- Optimization variables are voltages (squared magnitudes) v_j and power injections s_j respectively
- A current source or an impedance will introduce additional var and constraint.

Single-phase OPF

Power flow equations

- All lines point away from bus 0 (root)

$$\sum_{k:j\to k} S_{jk} = S_{ij} - z_{ij}\ell_{ij} + s_j, \qquad j \in \overline{N}$$

$$v_j - v_k = 2\,\mathrm{Re}\left(z_{jk}^{\mathsf{H}} S_{jk}\right) - |z_{jk}|^2 \ell_{jk}, \qquad j \to k \in E$$

$$v_j \ell_{jk} = |S_{jk}|^2, \qquad j \to k \in E$$

Operational constraints

$$s_j^{\min} \le s_j \le s_j^{\max}$$

$$v_j^{\min} \le v_j \le v_j^{\max}$$

$$\ell_{jk} \le I_{jk}^{\max}$$

Single-phase OPF

Feasible set

$$\mathbb{T}_0 := \left\{ x := (s, v, \ell, S) \in \mathbb{R}^{6N+3} \mid x \text{ satisfies PF equations \& operational constraints} \right\}$$

OPF in BFM

$$\min_{x \in \mathbb{T}_0} \quad C(x)$$

Single-phase OPF

Equivalence

Recall for BIM:

- Feasible set: $\mathbb{V} := \left\{ V \in \mathbb{C}^{N+1} \mid V \text{ satisfies operational constraints} \right\}$
- OPF: $\min_{V \in \mathbb{V}} \quad C_0(V)$

OPF in BFM is equivalent to OPF in BIM:

- Feasible sets $\mathbb{T}_0$ and $\mathbb{V}$ are equivalent (Ch 6)
- … provided cost functions $C(x)$ and $C_0(V)$ are the same

Three-phase OPF

Variables (x, u):

1. Directly generalizes vars in single-phase OPF ($\mathbb{S}_+^n$: complex psd matrices)

$$s_j \in \mathbb{C}^3, \qquad v_j \in \mathbb{S}_+^3, \qquad j \in \overline{N}$$
$$\ell_{jk} \in \mathbb{S}_+^3, \qquad S_{jk} \in \mathbb{C}^{3\times 3}, \qquad j \to k \in E$$

To write conversion rule for power sources, introduce phasors as additional vars

$$\left(V_j, j \in \overline{N}\right), \qquad \left(\tilde{I}_{jk}, j \to k \in E\right)$$

Let $x := (s, v, \ell, V, \tilde{I}, S)$

Three-phase OPF

Variables (x, u):

2. Internal variables $u := \left(u_j, j \in \overline{N}\right)$ of 3-phase devices

voltage source : $u_j := V_j^{Y/\Delta} \in \mathbb{C}^3$

power source : $u_j := \left(u_{j1}, u_{j2}\right) = \left(s_j^{Y/\Delta}, I_j^{Y/\Delta}\right) \in \mathbb{C}^6$

Three-phase OPF

Equality constraints

1. Power flow equations (from Ch 10):

$$\sum_{k:j\to k} \text{diag}(S_{jk}) = \text{diag}\left(S_{ij} - z_{ij}\ell_{ij}\right) + s_j, \qquad j \in \overline{N}$$

$$v_j - v_k = \left(z_{jk} S_{jk}^{\mathsf{H}} + S_{jk} z_{jk}^{\mathsf{H}}\right) - z_{jk}\,\ell_{jk}\, z_{jk}^{\mathsf{H}}, \qquad j \to k \in E$$

$$\begin{bmatrix} v_j & S_{jk} \\ S_{jk}^{\mathsf{H}} & \ell_{jk} \end{bmatrix} \succeq 0, \qquad j \to k \in E$$

$$\text{rank}\begin{bmatrix} v_j & S_{jk} \\ S_{jk}^{\mathsf{H}} & \ell_{jk} \end{bmatrix} = 1, \qquad j \to k \in E$$

$$v_j = V_j V_j^{\mathsf{H}}, \quad \ell_{jk} = \tilde{I}_{jk} \tilde{I}_{jk}^{\mathsf{H}}, \quad S_{jk} = V_j \tilde{I}_{jk}^{\mathsf{H}}, \qquad j \to k \in E$$

additional equations

Three-phase OPF

Equality constraints

1. Power flow equations (from Ch 10):

$$\sum_{k:j\to k} \text{diag}(S_{jk}) = \text{diag}\left(S_{ij} - z_{ij}\ell_{ij}\right) + s_j, \qquad j \in \overline{N}$$

$$v_j - v_k = \left(z_{jk} S_{jk}^{\mathsf{H}} + S_{jk} z_{jk}^{\mathsf{H}}\right) - z_{jk}\,\ell_{jk}\, z_{jk}^{\mathsf{H}}, \qquad j \to k \in E$$

$$\begin{bmatrix} v_j & S_{jk} \\ S_{jk}^{\mathsf{H}} & \ell_{jk} \end{bmatrix} \succeq 0, \qquad j \to k \in E$$

$$\text{rank} \begin{bmatrix} v_j & S_{jk} \\ S_{jk}^{\mathsf{H}} & \ell_{jk} \end{bmatrix} = 1, \qquad j \to k \in E$$

$$v_j = V_j V_j^{\mathsf{H}}, \quad \ell_{jk} = \tilde{I}_{jk}\tilde{I}_{jk}^{\mathsf{H}}, \quad S_{jk} = V_j \tilde{I}_{jk}^{\mathsf{H}}, \qquad j \to k \in E$$

redundant constraints kept for semidefinite relaxation (later)

Three-phase OPF

Equality constraints

2. Conversion rules for voltage & power sources (assume $\gamma_j^Y := V_j^n = 0$)

voltage source :

$$Y: \qquad v_j = V_j^Y V_j^{Y\mathsf{H}} = u_j u_j^{\mathsf{H}}$$

$$\Delta: \qquad \Gamma v_j \Gamma^{\mathsf{T}} = V_j^{\Delta} V_j^{\Delta\mathsf{H}} = u_j u_j^{\mathsf{H}}$$

power source :

$$Y: \qquad s_j = -\,\text{diag}\left(V_j u_{j2}^{\mathsf{H}}\right), \qquad s_j = -\,u_{j1}$$

$$\Delta: \qquad s_j = -\,\text{diag}\left(V_j u_{j2}^{\mathsf{H}} \Gamma\right), \qquad u_{j1} = \text{diag}\left(\Gamma V_j u_{j2}^{\mathsf{H}}\right)$$

Three-phase OPF

Inequality constraints

1. Operational constraints on x:

injection limits:	$s_j^{\min} \ \leq \ s_j \ \leq \ s_j^{\max},$	$j \in \overline{N}$
voltage limits:	$v_j^{\min} \ \leq \ \text{diag}\left(v_j\right) \ \leq \ v_j^{\max},$	$j \in \overline{N}$
line limits:	$\text{diag}\left(\ell_{jk}\right) \ \leq \ I_{jk}^{\max},$	$(j,k) \in E$

Three-phase OPF

Inequality constraints

2. Operational constraints on internal vars u_j:

voltage source: $v_j^{\phi n\,\min} \leq \left|V_j^{\phi n}\right|^2 \leq v_j^{\phi n\,\max}, \qquad v_j^{\phi\varphi\,\min} \leq \left|V_j^{\phi\varphi}\right|^2 \leq v_j^{\phi\varphi\,\max}$

power source: $s_j^{Y\,\min} \leq s_j^{Y} \leq s_j^{Y\,\max}, \qquad \left|I_j^{\phi n}\right|^2 \leq I_j^{\phi n\,\max}$

$s_j^{\Delta\,\min} \leq s_j^{\Delta} \leq s_j^{\Delta\,\max}, \qquad \left|I_j^{\phi\varphi}\right|^2 \leq I_j^{\phi\varphi\,\max}$

Three-phase OPF

Feasible set

$$\mathbb{T}_{3p} := \left\{ (x, u) := (s, v, \ell, V, \tilde{I}, S, u) \mid (x, u) \text{ satisfies all constraints} \right\}$$

OPF in BFM

$$\min_{(x,u)\in\mathbb{T}_{3p}} \quad C(x, u)$$

Three-phase OPF in BFM is equivalent to three-phase OPF in BIM:

- Their feasible sets are equivalent (Ch 10)
- … provided their cost functions are equivalent

Outline

1. Bus injection model
2. Branch flow model
3. OPF applications
   - Voltage control (distribution grid)
4. Optimization algorithms

Voltage control

Distribution system

Voltage instability: magnitudes fluctuate outside their limits

- PVs may push magnitudes above upper limits
- EVs may push magnitudes below lower limits

Traditional solution

- Infrastructure upgrade: more/larger transformers, wires, etc

Non-wire solution

- Distributed energy resources (DER) optimization
- e.g. batteries, smart inverters, demand response
- Can formulate as an OPF

Voltage control

Optimal battery operation

$$\min_{u,V,\, b} \quad \sum_t \sum_j \left(|V_j(t)|^2 - v_j^{\text{ref}}(t) \right)^2$$

deviation from nominal voltages

$$\text{s.t.} \quad u_j(t) + \sigma_j(t) = \sum_{k:j\sim k} S_{jk}(V(t)), \qquad \underline{v}_j \le |V_j(t)|^2 \le \overline{v}_j$$

$$|S_{jk}(V(t))| \le \overline{S}_{jk}, \qquad |S_{kj}(V(t))| \le \overline{S}_{kj}$$

Voltage control

Optimal battery operation

$$\min_{u, V, b} \quad \sum_t \sum_j \left(|V_j(t)|^2 - v_j^{\text{ref}}(t) \right)^2$$

deviation from nominal voltages

$$\text{s.t.} \quad u_j(t) + \sigma_j(t) = \sum_{k: j \sim k} S_{jk}(V(t)), \qquad \underline{v}_j \le |V_j(t)|^2 \le \overline{v}_j$$

$$|S_{jk}(V(t))| \le \overline{S}_{jk}, \qquad |S_{kj}(V(t))| \le \overline{S}_{kj}$$

$$b_j(t+1) = b_j(t) - \text{Re}\left(u_j(t) \right)$$

charging/discharging (100% efficiency)

Voltage control

Optimal battery operation

$$\min_{u,V,\,b} \quad \sum_t \sum_j \left(|V_j(t)|^2 - v_j^{\text{ref}}(t) \right)^2$$

deviation from nominal voltages

$$\text{s.t.} \quad u_j(t) + \sigma_j(t) = \sum_{k:j\sim k} S_{jk}(V(t)), \qquad \underline{v}_j \le |V_j(t)|^2 \le \overline{v}_j$$

$$|S_{jk}(V(t))| \le \overline{S}_{jk}, \qquad |S_{kj}(V(t))| \le \overline{S}_{kj}$$

$$b_j(t+1) = b_j(t) - \text{Re}\left(u_j(t)\right)$$

charging/discharging (100% efficiency)

$$\underline{u}_j \le \text{Re}\left(u_j(t)\right) \le \overline{u}_j, \qquad 0 \le b_j(t) \le B_j$$

power limit — energy limit

Voltage control

Optimal battery placement

$$\min_{u,V,\ b,\ B} \quad \sum_t \sum_j \left(|V_j(t)|^2 - v_j^{\text{ref}}(t) \right)^2 + \sum_j c_j B_j$$

$$\text{s.t.} \quad u_j(t) + \sigma_j(t) = \sum_{k:j\sim k} S_{jk}(V(t)), \qquad \underline{v}_j \leq |V_j(t)|^2 \leq \overline{v}_j$$

$$|S_{jk}(V(t))| \leq \overline{S}_{jk}, \qquad |S_{kj}(V(t))| \leq \overline{S}_{kj}$$

$$b_j(t+1) = b_j(t) - \text{Re}\left(u_j(t)\right)$$

$$\underline{u}_j \leq \text{Re}\left(u_j(t)\right) \leq \overline{u}_j, \qquad 0 \leq b_j(t) \leq B_j$$

$B_j^{\text{opt}} > 0$: place battery at bus j

Outline

1. Bus injection model
2. Branch flow model
3. Optimization algorithms
    - Newton-Raphson algorithm
    - Interior-point algorithm

Complex formulation

Even though OPF is often formulated in $\mathbb{C}$, it is converted to $\mathbb{R}$ before being solved iteratively

Example: QCQP

$$\min_{x\in\mathbb{C}^n} \quad x^{\mathsf{H}} C_0 x$$
$$\text{s.t.} \quad x^{\mathsf{H}} C_l x \ \le \ b_l, \qquad l = 1,\ldots,L$$

- $C_l : n \times n$ Hermitian matrix
- $b_l \in \mathbb{R}$

Equivalent to:

$$\min_{(x_r,x_i)\in\mathbb{R}^{2n}} \quad \begin{bmatrix} x_r \\ x_i \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} C_{0r} & -C_{0i} \\ C_{0i} & C_{0r} \end{bmatrix} \begin{bmatrix} x_r \\ x_i \end{bmatrix}$$
$$\text{s.t.} \quad \begin{bmatrix} x_r \\ x_i \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} C_{lr} & -C_{li} \\ C_{li} & C_{lr} \end{bmatrix} \begin{bmatrix} x_r \\ x_i \end{bmatrix} \le \ b_l, \quad l = 1,\ldots,L$$

- $2n \times 2n$ symmetric matrices

Algorithms for OPF

Popular algorithms

Newton-Raphson algorithm

- 2nd order algorithm
- Interior-point algorithm

Interior-point algorithm

- Based on barrier functions
- Uses of Newton-Raphson algorithm for subproblems

Newton-Raphson algorithm

NR is algorithm for solving

$$F(x) = 0, \qquad F : \mathbb{R}^n \to R^n$$

Iteratively:

$$x(t+1) = \; y(t) + \Delta x(t)$$

$$J(y(t))\, \Delta x(t) = -\, F(x(t))$$

where $J(x) := \frac{\partial F}{\partial x}(x)$ is Facobian of F

Application to optimization problems:

- $F(x) = 0$ is KKT condition
- If NR converges, it computes a KKT point x^{opt}
- x^{opt} is a global optimal if the problem is convex (feasible otherwise)

Newton-Raphson algorithm

Describe NR progressively for solving

- Linear equality constrained problems
- Nonlinear equality constrained problems
- Inequality constrained problems

Newton-Raphson algorithm

Linear equality constraint

Consider

$$\min_{x \in \mathbb{R}^n} \quad f(x) \qquad \text{s.t.} \qquad Ax \;=\; b$$

where

- $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is twice continuously differentiable
- $A \in \mathbb{R}^{m \times n}$

Newton-Raphson algorithm

Linear equality constraint

Consider

$$\min_{x\in\mathbb{R}^n} \quad f(x) \qquad \text{s.t.} \qquad Ax \;=\; b$$

Lagrangian:

$$L(x,\lambda) \;:=\; f(x) + \lambda^{\mathsf{T}}(Ax - b)$$

Jacobian of $L(x,\lambda)$:

$$F(x,\lambda) \;:=\; \begin{bmatrix} \nabla_x L(x,\lambda) \\ \nabla_\lambda L(x,\lambda) \end{bmatrix} \;=\; \begin{bmatrix} \nabla f(x) + A^{\mathsf{T}}\lambda \\ Ax - b \end{bmatrix}$$

KKT condition to be solved by NR algorithm:

$$F(x,\lambda) = 0$$

Newton-Raphson algorithm

Linear equality constraint

Consider

$$\min_{x \in \mathbb{R}^n} \quad f(x) \qquad \text{s.t.} \qquad Ax \;=\; b$$

Jacobian of $F(x, \lambda)$:

$$J(x,\lambda) \;=\; \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(x) & A^\top \\ A & 0 \end{bmatrix}$$

- KKT matrix
- Independent of λ

NR iteration:

$$\begin{bmatrix} x(t+1) \\ \lambda(t+1) \end{bmatrix} = \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} + \begin{bmatrix} \Delta x(t) \\ \Delta \lambda(t) \end{bmatrix} \quad \text{where} \quad \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(x(t)) & A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x(t) \\ \Delta \lambda(t) \end{bmatrix} = -\begin{bmatrix} \nabla f(x(t)) + A^\top \lambda(t) \\ Ax(t) - b \end{bmatrix}$$

Newton-Raphson algorithm

Nonlinear equality constraint

Consider

$$\min_{x \in \mathbb{R}^n} \quad f(x) \qquad \text{s.t.} \qquad g(x) = 0$$

where

- $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are twice continuously differentiable

Follow the same procedure as for linear equality constrained problems

Newton-Raphson algorithm

Nonlinear equality constraint

Consider

$$\min_{x \in \mathbb{R}^n} \quad f(x) \qquad \text{s.t.} \qquad g(x) = 0$$

Lagrangian:

$$L(x, \lambda) \ := \ f(x) + \lambda^{\mathsf{T}} g(x)$$

Jacobian of $L(x, \lambda)$:

$$F(x, \lambda) \ := \ \begin{bmatrix} \nabla_x L(x, \lambda) \\ \nabla_\lambda L(x, \lambda) \end{bmatrix} \ = \ \begin{bmatrix} \nabla f(x) + \frac{\partial g}{\partial x}(x)^{\mathsf{T}} \lambda \\ g(x) \end{bmatrix}$$

KKT condition to be solved by NR algorithm:

$$F(x, \lambda) = 0$$

Newton-Raphson algorithm

Nonlinear equality constraint

Consider

$$\min_{x \in \mathbb{R}^n} \quad f(x) \qquad \text{s.t.} \qquad g(x) = 0$$

Jacobian of $F(x, \lambda)$:

$$J(x,\lambda) \quad = \quad \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(x) + \sum_k \frac{\partial^2 g_k}{\partial x^2}\lambda_k & \frac{\partial g}{\partial x}(x)^{\mathsf{T}} \\ \frac{\partial g}{\partial x}(x) & 0 \end{bmatrix}$$

NR iteration:

$$\begin{bmatrix} x(t+1) \\ \lambda(t+1) \end{bmatrix} = \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} + \begin{bmatrix} \Delta x(t) \\ \Delta\lambda(t) \end{bmatrix} \quad \text{where} \quad J(x,\lambda)\begin{bmatrix} \Delta x(t) \\ \Delta\lambda(t) \end{bmatrix} = -\begin{bmatrix} \nabla f(x(t)) + \frac{\partial g}{\partial x}(x(t))^{\mathsf{T}}\lambda(t) \\ g(x(t)) \end{bmatrix}$$

Newton-Raphson algorithm

Inequality constraint

Consider

$$\min_{x \in \mathbb{R}^n} \quad f(x) \qquad \text{s.t.} \qquad g(x) \leq 0$$

where

- $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are twice continuously differentiable

Two common solution approaches

1. Introduce slack var $z \geq 0$ to reduce the inequality into a simple inequality constraint:

$$\min_{(x,z) \in \mathbb{R}^{n+m}} \quad f(x) \qquad \text{s.t.} \qquad g(x) + z \;=\; 0, \quad z \;\geq\; 0$$

2. Replace constraint by a penalty term and reduce to unconstrained problem:

$$\min_{x \in \mathbb{R}^n} \quad f(x) \;+\; \frac{1}{t}\phi(x)$$

This is the approach of interior-point algorithms !

Interior-point algorithm

Basic idea

Consider

$$\min_{x \in \mathbb{R}^n} \quad f_0(x) \qquad \text{s.t.} \qquad f(x) \leq 0, \quad g(x) = 0$$

where

- $f_0 : \mathbb{R}^n \to \mathbb{R}, f : \mathbb{R}^n \to \mathbb{R}^m, g : \mathbb{R}^n \to \mathbb{R}^p$ are twice continuously differentiable

Basic idea:

- Approximate problem by equality constrained problem by replacing $f(x) \leq 0$ by a barrier function
- Solve the approximate problem by Newton-Raphson methods

Interior-point algorithm

Log barrier function

Log barrier function $\phi : \mathbb{R}^n \to \mathbb{R}$ is

$$\phi(x) \;:=\; -\sum_{i=1}^{m} \log(-f_i(x))$$

over $\text{dom}\phi \;:=\; \{x \in \mathbb{R}^n : f_i(x) < 0, i = 1,\ldots,m\}$

Properties:

- $\phi(x) \to \infty$ as $f_i(x) \to 0$ for any i
- $\nabla \phi(x) \;=\; \sum_{i=1}^{m} \frac{1}{-f_i(x)} \nabla f_i(x)$
- $\frac{\partial^2 \phi}{\partial x^2}(x) \;=\; \sum_i \frac{1}{f_i^2(x)} \nabla f_i(x) \nabla f_i^{\mathsf{T}}(x) \;+\; \sum_i \frac{1}{-f_i(x)} \frac{\partial^2 f_i}{\partial x^2}(x)$

Interior-point algorithm

Approximate problem

Consider

$$\min_{x\in\mathbb{R}^n} \quad f_0(x) \qquad \text{s.t.} \qquad f(x) \ \leq \ 0, \quad g(x) \ = \ 0$$

Approximate problem

$$\min_{x\in\mathbb{R}^n} \quad f_0(x) + \frac{1}{t}\phi(x) \qquad \text{s.t.} \qquad g(x) \ = \ 0$$

or

Problem(t) : $$\min_{x\in\mathbb{R}^n} \quad tf_0(x) + \phi(x) \qquad \text{s.t.} \qquad g(x) \ = \ 0$$

- Larger $t > 0 \implies$ more accurate approximation

Barrier method

A popular interior-point method

Basic idea

- Solve Problem(t) for an increasing sequence of $t > 0$ until solution is accurate enough
- For each t, solve Problem(t) using Newton-Raphson algorithm

Questions

- How to choose the sequence of t?
- When to terminate?

Answer these question for convex problems

Barrier method

Assumptions

1. Original problem is convex, i.e., $f_0, f_1, \ldots, f_m$ are convex and $g(x) = Ax - b$

2. For each $t > 0$, Newton-Raphson algorithm converges to the unique optimal solution $x(t)$ of the approximate problem

- Central point : optimal solution $x(t)$
- Central path : set $\{x(t) : t > 0\}$ of central points

Barrier method

Central point $x(t)$

1. Original problem is convex, i.e., $f_0, f_1, \ldots, f_m$ are convex and $g(x) = Ax - b$
2. For each $t > 0$, Newton-Raphson algorithm converges to the unique optimal solution $x(t)$ of the approximate problem

Theorem

For each $t > 0$

1. $x(t)$ is feasible for original problem
2. Objective value is at most m/t away from optimal value, i.e., $f_0(x(t)) - f_0^{\text{opt}} \leq \frac{m}{t}$

   In particular $f_0(x(t)) \to f_0^{\text{opt}}$ as $t \to \infty$

Barrier method

Input: *strictly* feasible x, initial $t := t_0$, scaling factor $\gamma > 1$, tolerance ε.
Output: an approximate solution x

1. **while** $t \leq \frac{m}{\varepsilon}$ **do**
   (a) Solve Problem(t) to compute $x(t)$ using the Newton-Raphson algorithm starting from x.
   (b) $x \leftarrow x(t)$.
   (c) $t \leftarrow \gamma t$.
2. **Return**: x.

In principle, one can solve Problem(t) with $t := m/\epsilon$ instead of solving a sequence of Problem(t).
In practice, barrier method works better.