Power Systems Analysis

Chapter 6 Branch flow models

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Outline

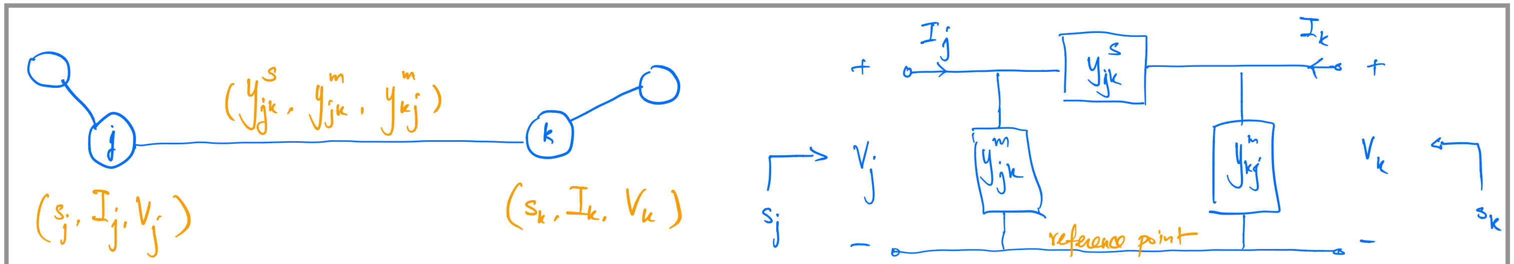
1. General network
2. Radial network
3. Equivalence
4. Backward forward sweep
5. Linearized model

Outline

1. General network
   - Complex form power flow equations
   - Real form power flow equations
2. Radial network
3. Equivalence
4. Backward forward sweep
5. Linearized model

General network

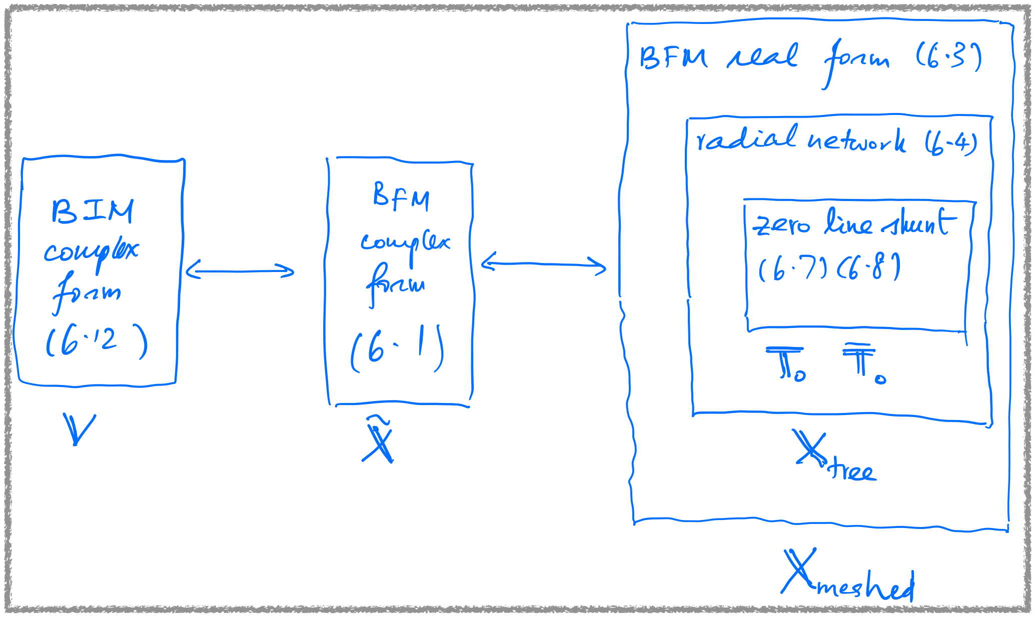
1. Network $G := (\overline{N}, E)$
   - $\overline{N} := \{0\} \cup N := \{0\} \cup \{1,\ldots,N\}$: buses/nodes
   - $E \subseteq \overline{N} \times \overline{N}$: lines/links/edges
2. Each line (j,k) is parameterized by $\left(y^s_{jk}, y^m_{jk}, y^m_{kj}\right)$
   - y^s_{jk} : series admittance
   - y^m_{jk}, y^m_{kj} : shunt admittances, generally different



(a) Graph representation (b) Π equivalent circuit

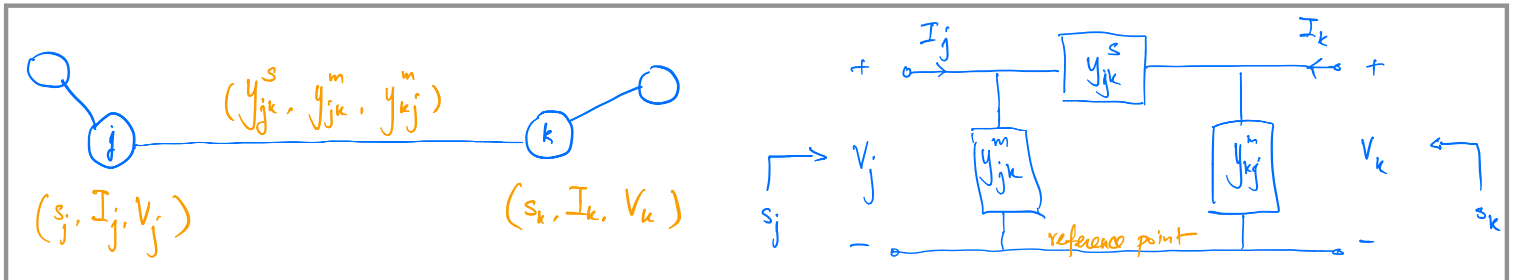
General network

1. We will introduce several BFM models
    - General: complex form, real form
    - Radial: with/without shunt admits.
2. Each model defined by
    - Set of variables
    - Set of power flow equations relating these variables
3. These models are equivalent to each other, and to BIM



General network

Branch currents



(a) Graph representation (b) Π equivalent circuit

Sending-end currents

$$I_{jk} \;=\; y^s_{jk}(V_j - V_k) \;+\; y^m_{jk}\, V_j, \qquad I_{kj} \;=\; y^s_{jk}(V_k - V_j) \;+\; y^m_{kj}\, V_k,$$

Bus injection model: relate nodal variables s and V

$$s_j = \sum_{k:j\sim k} \left(y^s_{jk}\right)^H \left(|V_j|^2 - V_j V_k^H\right) \;+\; \left(y^m_{jj}\right)^H |V_j|^2$$

General network

Complex form

Branch flow model: includes branch vars as well

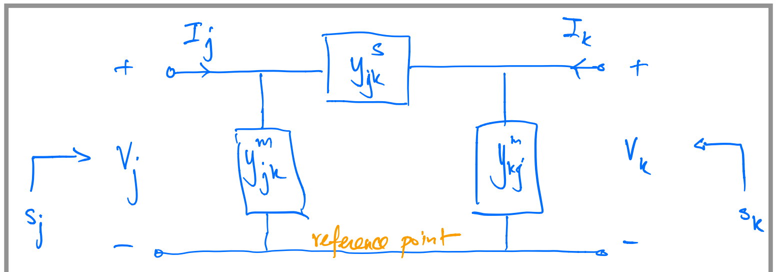
- Branch currents $\left(I_{jk}, I_{kj}\right)$
- Branch power $\left(S_{jk}, S_{kj}\right)$

$$s_j = \sum_{k:j\sim k} S_{jk}$$

$$S_{jk} = V_j I_{jk}^H, \qquad S_{kj} = V_k I_{kj}^H$$

$$I_{jk} = y_{jk}^s (V_j - V_k) + y_{jk}^m V_j$$

$$I_{kj} = y_{kj}^s (V_k - V_j) + y_{kj}^m V_k$$



(b) Π equivalent circuit

This model is equivalent to BIM (later)

- Serves as a bridge to BIM

General network

Real form

Key feature of original Dist Flow equations (branch flow model) of Baran-Wu1989

- No voltage/current phase angles
- Suitable for radial networks (tree topology)
- We generalize to meshed networks

For each bus j

- $s_j := (p_j, q_j)$ or $s_j := p_j + iq_j$: power injection
- v_j : squared voltage magnitude

For each branch (j, k)

- $S_{jk} := \left(P_{jk}, Q_{jk}\right)$ or $S_{jk} := P_{jk} + iQ_{jk}$: sending-end power $j \to k$; also S_{kj} from $k \to j$
- $\left(\ell_{jk}, \ell_{kj}\right)$: squared magnitude of sending-end current $j \to k$, and $k \to j$

General network

Real form

The variables v_i and $\left(\ell_{jk}, \ell_{kj}\right)$ contain no angle information

Angle info must be recoverable from a power flow solution $x := (s, v, \ell, S) \in \mathbb{R}^{3(N+1)+6M}$

- This is easy for radial networks
- Trickier for meshed networks

General network

Real form

For each line (j,k) let:

$$z_{jk}^s := \left(y_{jk}^s\right)^{-1} =: z_{kj}^s$$

$$\alpha_{jk} := 1 + z_{jk}^s\, y_{jk}^m, \qquad \alpha_{kj} := 1 + z_{kj}^s\, y_{kj}^m$$

$\alpha_{jk} = \alpha_{kj}$ if and only if $y_{jk}^m = y_{kj}^m$

$\alpha_{jk} = \alpha_{kj} = 1$ if and only if $y_{jk}^m = y_{kj}^m = 0$

General network

Real form

For each line (j,k) let:

$$z^s_{jk} := \left(y^s_{jk}\right)^{-1} =: z^s_{kj}$$

$$\alpha_{jk} := 1 + z^s_{jk}\, y^m_{jk}, \qquad \alpha_{kj} := 1 + z^s_{kj}\, y^m_{kj}$$

Given $x := (s, v, \ell, S) \in \mathbb{R}^{3(N+1)+6M}$ define nonlinear functions:

$$\beta_{jk}(x) := \angle\left(\alpha^H_{jk}\, v_j - \left(z^s_{jk}\right)^H S_{jk}\right)$$

$$\beta_{kj}(x) := \angle\left(\alpha^H_{kj}\, v_k - \left(z^s_{jk}\right)^H S_{kj}\right)$$

If x is a power flow solution, then $\left(\beta_{jk}(x), \beta_{kj}(x)\right)$ are angle differences across (j,k)

General network

Real form

$$s_j = \sum_{k: j \sim k} S_{jk}$$

power balance

General network

Real form

$$s_j = \sum_{k:j\sim k} S_{jk}$$ power balance

$$\left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$ branch power magnitude

The complex notation is only shorthand for real equations

$$p_j = \sum_k P_{jk}, \qquad q_j = \sum_k Q_{jk}$$

$$v_j \ell_{jk} = P_{jk}^2 + Q_{jk}^2, \qquad v_k \ell_{kj} = P_{kj}^2 + Q_{kj}^2$$

General network

Real form

$$s_j = \sum_{k:j\sim k} S_{jk}$$ power balance

$$\left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$ branch power magnitude

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^H S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$ Ohm's law, KCL (magnitude)

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^H S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$

General network

Real form

$$s_j = \sum_{k:j\sim k} S_{jk}$$ power balance

$$\left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$ branch power magnitude

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^H S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$ Ohm's law, KCL (magnitude)

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^H S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$

there exists $\theta \in \mathbb{R}^{N+1}$ s.t. $\beta_{jk}(x) = \theta_j - \theta_k$ cycle condition

$$\beta_{kj}(x) = \theta_k - \theta_j$$

General network

Real form

Cycle condition on x is highly nonlinear

$$\beta_{jk}(x) := \angle\left(\alpha_{jk}^H v_j - \left(z_{jk}^s\right)^H S_{jk} \right)$$

$$\beta_{kj}(x) := \angle\left(\alpha_{kj}^H v_k - \left(z_{jk}^s\right)^H S_{kj} \right)$$

$$\beta(x) = \begin{bmatrix} C^T \\ -C^T \end{bmatrix} \theta \text{ for some } \theta \in \mathbb{R}^{N+1}$$

It ensures angle consistency of a power flow solution x

General network

Real form

Any $x := (s, v, \ell, S) \in \mathbb{R}^{3(N+1)+6M}$ that satisfies power flow equations with

$$v \geq 0, \quad \ell \geq 0$$

is a power flow solution

Branch flow models have been most useful for radial networks

All BFMs for radial networks are special cases of this model

- Tree topology
- Tree topology with zero shunt admittances $y_{jk}^m = y_{kj}^m = 0$
- Tree topology with linear approximations

Outline

1. General network
2. Radial network
    - With shunt admittances
    - Without shunt admittances
3. Equivalence
4. Backward forward sweep
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Radial network

Cycle condition

Major simplification for radial network: nonlinear cycle condition becomes linear in x

$$\boxed{\begin{aligned} \beta_{jk}(x) &:= \angle\left(\alpha_{jk}^H v_j - \left(z_{jk}^s\right)^H S_{jk}\right) \\ \beta_{kj}(x) &:= \angle\left(\alpha_{kj}^H v_k - \left(z_{jk}^s\right)^H S_{kj}\right) \\ \beta(x) &= \begin{bmatrix} C^T \\ -C^T \end{bmatrix} \theta \ \text{ for some } \theta \in \mathbb{R}^{N+1} \end{aligned}}$$

general network

$$\Rightarrow$$

$$\boxed{\alpha_{jk}^H v_j - \left(z_{jk}^s\right)^H S_{jk} \ = \ \left(\alpha_{kj}^H v_k - \left(z_{kj}^s\right)^H S_{kj}\right)^H}$$

radial network

Radial network

With shunt admittances

$$s_j = \sum_{k:j\sim k} S_{jk}$$ power balance

$$\left|S_{jk}\right|^2 = v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 = v_k \ell_{kj}$$ branch power magnitude

$$\left|\alpha_{jk}\right|^2 v_j - v_k = 2\,\mathrm{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^H S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}$$ Ohm's law, KCL (magnitude)

$$\left|\alpha_{kj}\right|^2 v_k - v_j = 2\,\mathrm{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^H S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}$$

$$\boxed{\alpha^H_{jk} v_j - \left(z^s_{jk}\right)^H S_{jk} = \left(\alpha^H_{kj} v_k - \left(z^s_{kj}\right)^H S_{kj}\right)^H}$$ cycle condition

$2(N+1) + 6M$ real equations in $3(N+1) + 6M$ real vars $(M = N)$

Radial network

With shunt admittances

Any $x := (s, v, \ell, S) \in \mathbb{R}^{3(N+1)+6M}$ that satisfies power flow equations with

$$v \geq 0, \quad \ell \geq 0$$

is a power flow solution

All equations are linear in x, except the quadratic equalities

$$\left|S_{jk}\right|^2 \;=\; v_j \ell_{jk}, \qquad \left|S_{kj}\right|^2 \;=\; v_k \ell_{kj}$$

This can be relaxed to second-order cone constraint in OPF (later)

General network

Angle recovery

Treat network $G := (\overline{N}, E)$ as directed graph with arbitrary orientation

- (Re-)Define branch variables $\left(S_{jk}, \ell_{jk}\right)$ only in direction of lines (j, k)
- (Re-)Define $\beta(x) := \left(\beta_{jk}(x), (j,k) \in E\right)$ where

$$\beta_{jk}(x) := \angle\left(\alpha_{jk}^H v_j - \left(z_{jk}^s\right)^H S_{jk}\right)$$

Any power flow solution $x := (s, v, \ell, S) \in \mathbb{R}^{3(N+1)+6M}$ implies

$$\beta(x) = C^T\theta \text{ for some } \theta \in \mathbb{R}^{N+1}$$

Angle recovery:

$$V_j = \sqrt{v_j}\, e^{i\theta_j}, \qquad I_{jk} = \sqrt{\ell_{jk}}\, e^{i\left(\theta_j - \angle S_{jk}\right)}$$

Radial network

Without shunt admittances

When shunt admittances $y_{jk}^m = y_{kj}^m = 0$

- $\alpha_{jk} = \alpha_{kj} = 1$
- $\ell_{kj} = \ell_{jk}$ and $S_{kj} \ + \ S_{jk} \ = \ z_{jk}^s \ell_{jk}$

Can use directed graph with vars $\left(\ell_{jk}, S_{jk}\right)$ defined only in direction of lines (j, k)

Substitute $\left(\ell_{kj}, S_{kj}\right)$ in terms of $\left(\ell_{jk}, S_{jk}\right)$ into previous power flow equations yields original DistFlow equations of [Baran-Wu 1989]

Radial network

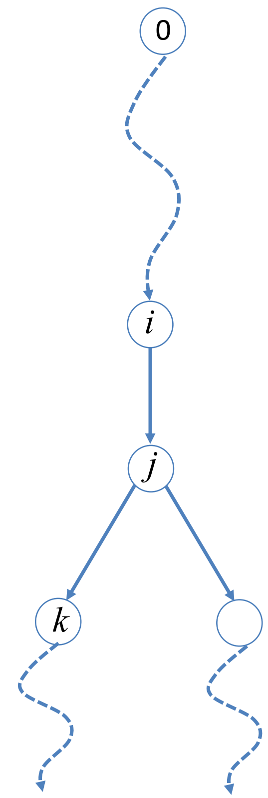
Without shunt admittances

DistFlow equations [Baran-Wu 1989] (down direction):

$$\sum_{k:j\to k} S_{jk} = S_{ij} - z_{ij}\ell_{ij} + s_j$$ power balance

$$v_j - v_k = 2\,\mathrm{Re}\left(z_{jk}^H S_{jk}\right) - |z_{jk}|^2 \ell_{jk}$$ Ohm's law, KCL (magnitude)

$$v_j \ell_{jk} = |S_{jk}|^2$$ branch power magnitude



All lines point away from bus 0

$2(N+1) + 2M$ real equations in $3(N+1) + 3M$ real vars $(M = N)$

- Given $\left(v_0, s_j, j \in N\right)$, there are $4N+2$ equations in $4N+2$ vars $\left(s_0, v_j, j \in N, \ell, S\right)$

Radial network

Without shunt admittances

DistFlow equations [Baran-Wu 1989] (down direction):

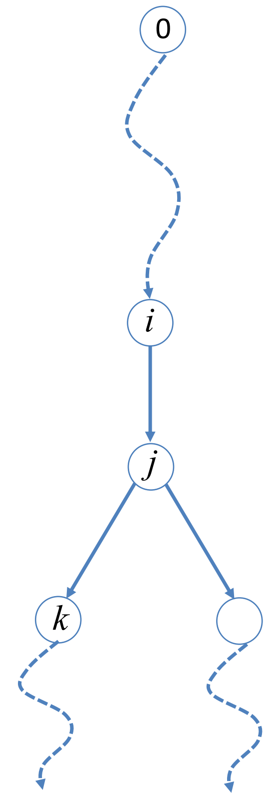
$$\sum_{k:j\to k} S_{jk} = S_{ij} - z_{ij}\ell_{ij} + s_j$$ power balance

$$v_j - v_k = 2\,\text{Re}\left(z_{jk}^H S_{jk}\right) - |z_{jk}|^2 \ell_{jk}$$ Ohm's law, KCL (magnitude)

$$v_j \ell_{jk} = |S_{jk}|^2$$ branch power magnitude

All equations are linear in x, except the quadratic equalities

$$v_j \ell_{jk} = \left|S_{jk}\right|^2$$



All lines point away from bus 0

Outline

1. General network
2. Radial network
3. Equivalence
4. Backward forward sweep
5. Linearized model

Equivalence

Recap

Bus injection model

- General networks: complex form, polar form, Cartesian form

Branch flow model

- General networks: complex form, real form
- Radial networks: with / without shunt admittances

All these models are equivalent

- In what sense?
- They consist of different equations with different variables in different domains

Equivalence
Solution set

Bus injection model

$$s_j = \sum_{k:j\sim k} \left(y^s_{jk}\right)^H \left(|V_j|^2 - V_j V_k^H \right) + \left(y^m_{jj}\right)^H |V_j|^2$$

Solution set

$$\mathbb{V} := \{(s, V) \in \mathbb{C}^{2(n+1)} \mid V \text{ satisfies BIM}\}$$

Equivalence
Solution set

Branch flow models: solution sets

$$\tilde{\mathbb{X}} := \{\tilde{x} : (s, V, I, S) \in \mathbb{C}^{2(N+1)+4M} \mid \tilde{x} \text{ satisfies BFM complex}\}$$

$$\mathbb{X}_{\text{meshed}} := \{x : (s, v, \ell, S) \in \mathbb{R}^{3(N+1)+6M)} \mid x \text{ satisfies BFM real}\}$$

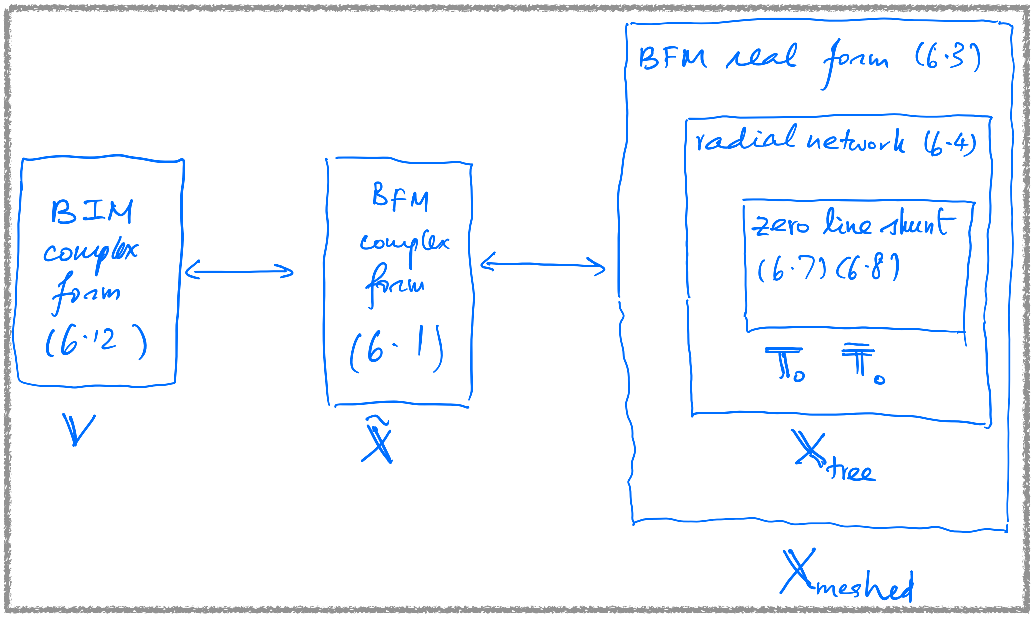
$$\mathbb{X}_{\text{tree}} := \{x : (s, v, \ell, S) \in \mathbb{R}^{9N+3} \mid x \text{ satisfies BFM radial}\}$$

$$\mathbb{T}_0 := \{x : (s, v, \ell, S) \in \mathbb{R}^{6N+3} \mid x \text{ satisfies BFM radial zero } y_{jk}^m\}$$

<u>Definition</u>: Two sets A and B are equivalent $(A \equiv B)$ if there is a bijection between them

Equivalence

Solution set



Theorem

Suppose G is connected

1. $\mathbb{V} \equiv \tilde{\mathbb{X}} \equiv \mathbb{X}_{\text{meshed}}$
2. If G is a tree, then $\mathbb{X}_{\text{meshed}} \equiv \mathbb{X}_{\text{tree}}$
3. If G is a tree and $y_{jk}^m = y_{kj}^m = 0$, then $\mathbb{X}_{\text{tree}} \equiv \mathbb{T}_0 \equiv \hat{\mathbb{T}}_0$

Equivalence

Bus injection models and branch flow models are equivalent

- Any result proved in one model holds also in another model

Some results are easier to formulate / prove in one model than the other

- BIM: semidefinite relaxation of OPF (later)
- BFM: some exact relation proofs

Should freely use whichever is more convenient for problem at hand

BFM is particularly suitable for modeling distribution systems

- Tree topology allows efficient computation of power flows (BFS)
- Seems to be much more numerically stable than BIM for large networks
- Models and relaxations extend to unbalanced 3ϕ networks

Outline

1. General network
2. Radial network
3. Equivalence
4. Backward forward sweep
    - For radial networks
5. Linearized model

Backward forward sweep

Efficient solution method for power flow equations

- Applicable for radial networks

Partition solution (x, y) into two groups of variables x and y

Each round of spatial iteration consists of a backward sweep and a forward sweep

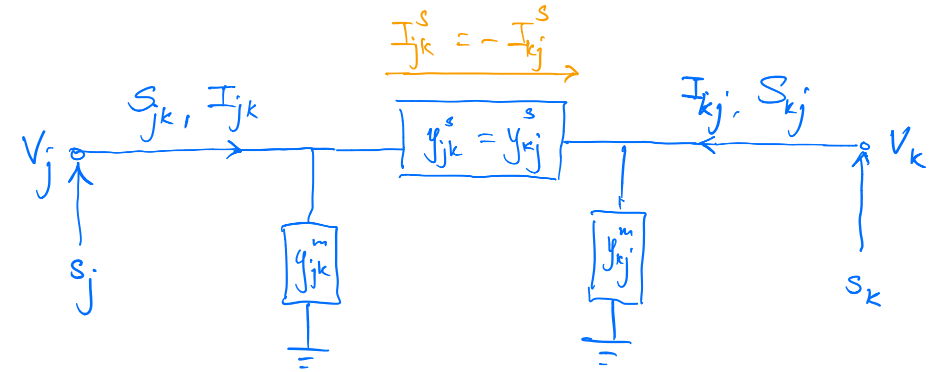
- Given y, compute each component x_j iteratively from leafs to root (backward)
- Given x, compute each component y_j iteratively from root to leaves (forward)

Iterate until stopping criterion

Different BFS methods differ in how to partition variables into x and y and the associated power flow equations

Backward forward sweep

Example



Use complex form BFM

Given: V_0 and $s := \left(s_j, j \in N\right)$

Compute: $V := \left(V_j, j \in N\right)$ and currents $I^s := \left(I^s_{jk}, (j,k) \in E\right)$ through series impedance

- All other variables $I_{jk} = I^s_{jk} + y^m_{jk} V_j$, I_{kj}, S_{jk}, S_{kj} can then be computed
- Can also compute V_j and I_{jk} instead (exercise)
- Advantage: $I^s_{jk} = -\, I^s_{kj}$

D. Shirmohammadi, H. W. Hong, A. Semlyen, and G. X. Luo. A compensation-based power flow method for weakly meshed distribution and transmission networks. *IEEE Transactions on Power Systems*, 3(2):753–762, May 1988.

Backward forward sweep
Example

Power flow equation

$$s_j = V_j I_{ji}^H + \sum_{k:j\to k} V_j I_{jk}^H = V_j \left(\left(I_{ji}^s + y_{ji}^m V_j \right)^H + \sum_{k:j\to k} \left(I_{jk}^s + y_{jk}^m V_j \right)^H \right)$$

Substitute $I_{kj}^s = -I_{jk}^s$ to write all vars in direction of line $j \to k$:

$$\left(\frac{s_j}{V_j} \right)^H = -I_{ij}^s + y_{jj}^m V_j + \sum_{k:j\to k} I_{jk}^s$$

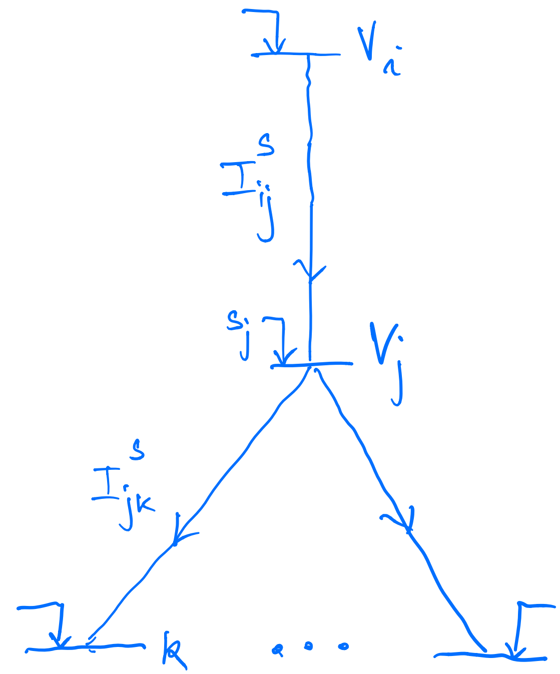
where $y_{jj}^m := \sum_k y_{jk}^m$

Backward forward sweep

Example

Rewrite in spatially recursive structure

$$I_{ij}^s = \sum_{k:j\to k} I_{jk}^s \; - \; \left(\left(\frac{s_j}{V_j}\right)^H - y_{jj}^m V_j \right)$$



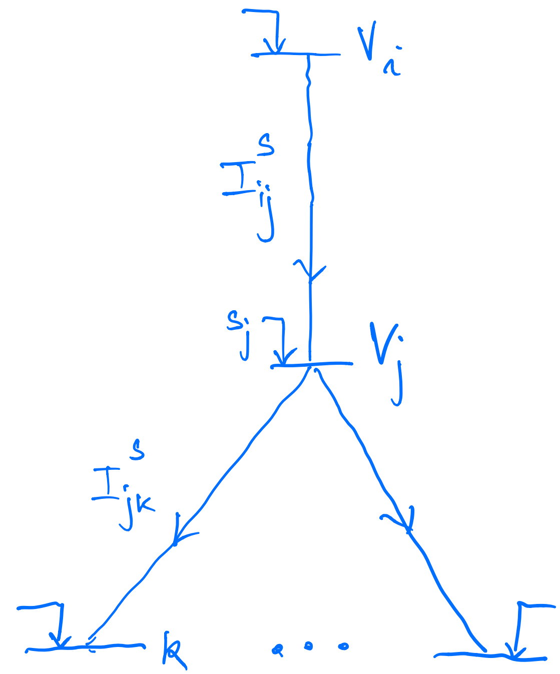
<u>Spatial iteration</u>: propagating from leafs towards root (bus 0) in reverse BFS order

- Given all voltages $V := \left(V_j, j \in \overline{N}\right)$
- Given all currents I_{jk}^s in previous layer
- Compute currents I_{ij}^s in current layer

Backward forward sweep
Example

Write in spatially recursive structure

$$V_j \;=\; V_i \;-\; z_{ij}^s I_{ij}^s,$$



Spatial iteration: propagating from root (bus 0) towards leafs in BFS order

- Given all currents $I^s := \left(I_{ij}^s, (i,j) \in E\right)$
- Given all voltages V_i in previous layer
- Compute voltages V_j in current layer

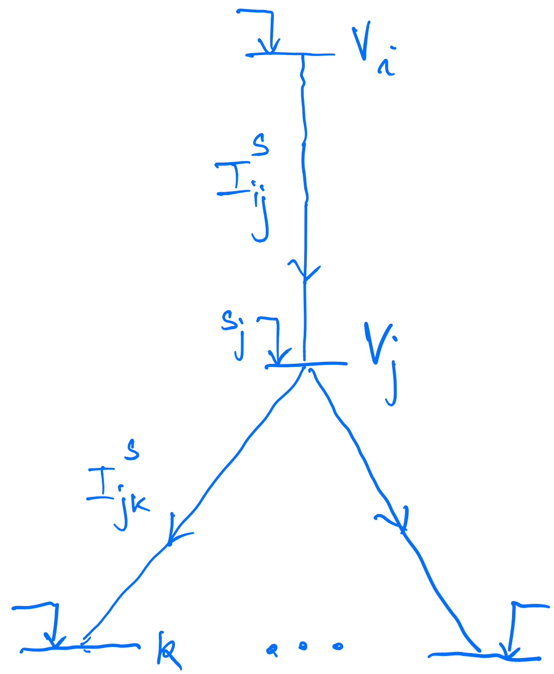
Backward forward sweep

Summary

Input: voltage $V_0 = 1$ pu and injections $(s_i, i \in N)$.
Output: currents $(I^s_{jk}, j \to k \in E)$ and voltages $(V_i, i \in N)$.

1. *Initialization.*

   - $V_0(t) := 1$ pu at bus $j = 0$ for all iterations $t = 1, 3, \ldots$.
   - $V_j(0) := 1$ pu at all buses $j \in N$ for iteration $t = 0$.



Backward forward sweep

Summary

2. Backward forward sweep: iterate $t = 1,3,5,\ldots$ till stopping criterion

(a) *Backward sweep.* Starting from the leaf nodes and working towards bus 0, compute

$$I^s_{ij}(t) \quad \leftarrow \quad \sum_{k:j\to k} I^s_{jk}(t) \quad - \quad \left(\left(\frac{s_j}{V_j(t-1)} \right)^H - y^m_{jj} V_j(t-1) \right), \qquad i \to j \in E$$

(b) *Forward sweep.* Starting front bus 0 and working towards the leaf nodes, compute

$$V_j(t+1) \quad = \quad V_i(t+1) \quad - \quad z^s_{ij} I^s_{ij}(t), \qquad j \in N$$

3. Output: $\quad I^s_{jk} := I^s_{jk}(t), \quad V_i := V_i(t+1)$

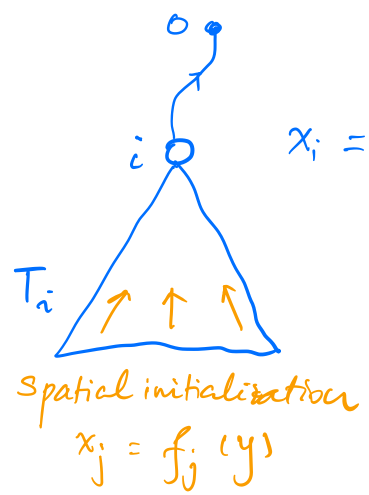
Backward forward sweep
General formulation

Backward sweep: let

- $\mathsf{T}_i^\circ :=$ {buses in subtree rooted at i, excluding i}
- $x_{\mathsf{T}_i^\circ} := \left(x_j, j \in \mathsf{T}_i^\circ\right)$

x satisfies a spatially recursive structure if

$$x_i \;=\; f_i\left(x_{\mathsf{T}_i^\circ};\; y\right)$$



$x_i = f_i(x_{T_i^\circ}; y)$

Backward forward sweep

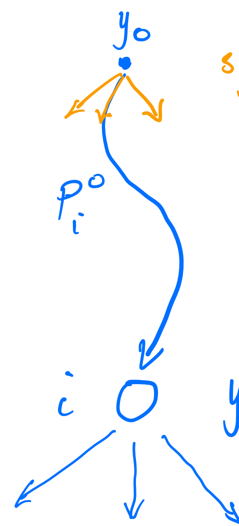
General formulation

Forward sweep: let

- $\mathsf{P}_i^\circ := \{$buses in path from root to i, inc. 0 but exc. $i\}$
- $y_{\mathsf{P}_i^\circ} := \left(y_j, j \in \mathsf{P}_i^\circ\right)$

y satisfies a spatially recursive structure if

$$y_i \quad = \quad g_i\left(y_{\mathsf{P}_i^\circ};\ x\right)$$



spatial initialization

$y_j = g_j(y_0; x)$

$y_i = g_i(y_{P_i^o};\ x)$

Backward forward sweep

General formulation

while *stopping criterion not met* **do**

(a) $t \leftarrow t+1;\ y_0(t) \leftarrow y_0;$

(b) *Backward sweep*: **for** *i starting from the leaf nodes and iterating towards bus 0* **do**

$$x_i(t) \quad \leftarrow \quad f_i\left(x_{\mathsf{T}_i^\circ}(t);\ y(t-1)\right), \qquad i \in \overline{N}$$

-

(c) *Forward sweep*: **for** *i starting front bus 0 and iterating towards the leaf nodes* **do**

$$y_i(t+1) \quad \leftarrow \quad g_i\left(y_{\mathsf{P}_i^\circ}(t+1);\ x(t)\right), \qquad i \in N$$

Backward forward sweep

Open question

Convergence analysis

Outline

1. General network
2. Radial network
3. Equivalence
4. Backward forward sweep
5. Linearized model
    - Analytical solution
    - Bounds on nonlinear solutions
    - Application: decentralized volt/var control

Linearized model

Linear DistFlow equations

For radial networks

Set $y^m_{jk} = y^m_{kj} = 0, \ \ell_{jk} = 0$

Linear DistFlow equations [Baran-Wu 1989]

$$\sum_{k:j\to k} S_{jk} = \sum_{i:i\to j} S_{ij} \ + \ s_j$$

$$v_j - v_k = \ 2\,\mathrm{Re}\left(z^H_{jk} S_{jk}\right)$$

Linearized model

Linear DistFlow equations

In vector form:

bus-by-line incidence matrix

$$C_{jl} \;=\; \begin{cases} 1 & \text{if } l = j \to k \text{ for some bus } k \\ -1 & \text{if } l = i \to j \text{ for some bus } i \\ 0 & \text{otherwise} \end{cases}$$

Linear DistFlow equations

$$s \;=\; C\,S$$
$$C^T v \;=\; 2\left(D_r P + D_x Q\right)$$

where $D_r := \text{diag}(r_l, l \in E), \quad D_x := \text{diag}(x_l, l \in E)$

$$\sum_{k:j\to k} S_{jk} \;=\; \sum_{i:i\to j} S_{ij} \;+\; s_j$$
$$v_j - v_k \;=\; 2\,\text{Re}\left(z_{jk}^H S_{jk}\right)$$

Linearized model

Linear DistFlow equations

Linear DistFlow can be solved explicitly

Given: $v_0 \ = \ 1$ pu, injection $\hat{s} := \left(s_j, j \in N\right)$

Determine: line power $S := \left(S_{jk}, j \to k \in E\right)$, voltage $\hat{v} := \left(v_j, j \in N\right)$, injection s_0

Key observation: Reduced incidence matrix has full rank

G connected $\implies$ $(N+1) \times N$ incidence matrix C has rank N

Decompose $C =: \begin{bmatrix} -c_0^T- \\ \hat{C} \end{bmatrix}$

G tree topology $\implies$ $N \times N$ reduced incidence matrix $\hat{C}$ is invertible

Linearized model

Linear DistFlow solution

Linear DistFlow:

$$\hat{s} = \hat{C}\,S$$

$$v_0\,c_0 \;+\; \hat{C}^T\hat{v} = 2\left(D_r P \;+\; D_x Q\right)$$

and $s_0 = c_0^T S$

$$s = C\,S$$
$$C^T v = 2\left(D_r P + D_x Q\right)$$

Solution:

$$S = \hat{C}^{-1}\hat{s}$$

$$\hat{v} = v_0\,\mathbf{1} \;+ 2\left(R\hat{p} + X\hat{q}\right)$$

voltages = v_0 + correction term $\left(\hat{p}, \hat{q}\right)$

where $R := \hat{C}^{-T} D_r \hat{C}^{-1}, \quad X := \hat{C}^{-T} D_x \hat{C}^{-1}$ are positive definite matrices

Linearized model

Bounds on nonlinear solution

Corollary

Fix v_0 and injections $\hat{s} \in \mathbb{R}^{2N}$ at non-reference buses. Let (v, ℓ, S) and $\left(v^{\text{lin}}, \ell^{\text{lin}}, S^{\text{lin}}\right)$ be a solution of nonlinear and linear DistFlow equations respectively (in the down direction).

1. $S_{ij} \geq S_{ij}^{\text{lin}}$
2. $v_j \leq v_j^{\text{lin}}$

Linear DistFlow ignores line losses and underestimates required power to supply loads

Outline

1. General network
2. Radial network
3. Equivalence
4. Backward forward sweep
5. Linearized model
    - Analytical solution
    - Bounds on nonlinear solutions
    - Application: decentralized volt/var control

Linearized model

Application: volt/var control

Volv/var control: control reactive power injections q to stabilize voltages $\hat{v}$

How should q adapt as voltages fluctuate?

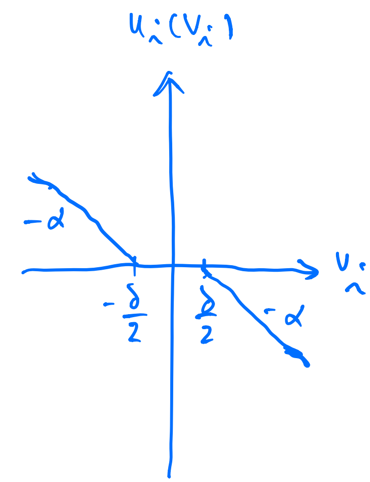
Linearized model
Application: volt/var control

Local memoryless feedback control:

$$q_j(t+1) = \left[u_j \left(v_j(t) - v_j^{\text{ref}} \right) \right]_{U_j}$$

$U_j := \{q_j : \underline{q}_j \le q_j \le \overline{q}_j\}$

Adapt reactive power $q_i(t)$ to drive voltage $v_j(t)$ towards target v_j^{ref}



Control $q_j(t+1)$ depends only on:

- Feedback: measured system state $v(t)$
- Memoryless: latest voltage $v(t)$ at time t, not history $v(s)$, $s < t$
- Local: local voltage $v_j(t)$ at bus j, not other voltages $v_k(t)$

Linearized model
Application: volt/var control

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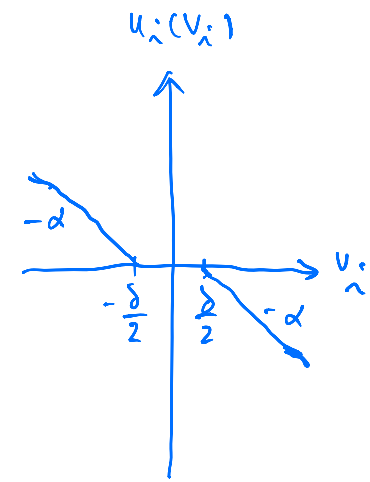

How does the closed-loop system behave ?
- under this simple local control
- if network is described by Linear DistFlow

Linearized model

Application: volt/var control

Linear DistFlow model describes how voltages (linearly) depend on control q:

$$v(q) = v_0 \mathbf{1} + 2\left(Rp + Xq\right) = 2Xq + \tilde{v}$$

where $\tilde{v} := v_0 \mathbf{1} + 2Rp$

Since

$$\frac{\partial v_j}{\partial q_j} = 2X_{jj} = \sum_{(i,k)\in \mathbb{P}_j} x_{ik} > 0$$

it justifies choosing u_j to be a decreasing function of $v_j(t) - v_j^{\text{ref}}$

Linearized model

Application: volt/var control

Assume measured voltage is given by Linear DistFlow, i.e., $v_j(t) = v_j(q(t))$

Closed-loop system is discrete-time dynamical system:

$$q_j(t+1) = \left[u_j \left(v_j(q(t)) - v_j^{\text{ref}} \right) \right]_{U_j}$$

where $v(q) \ = \ 2Xq \ + \tilde{v}$

Definition: q^* is a fixed point or equilibrium point if $q^* = \left[u \left(v(q^*) - v^{\text{ref}} \right) \right]_{U_j}$

Linearized model

Application: volt/var control

Assume measured voltage is given by Linear DistFlow, i.e., $v_j(t) = v_j(q(t))$

Closed-loop system is discrete-time dynamical system:

$$q_j(t+1) = \left[u_j \left(v_j(q(t)) - v_j^{\text{ref}} \right) \right]_{U_j}$$

where $v(q) \;=\; 2Xq \;+ \tilde{v}$

What are convergence and optimality properties of closed-loop system ?

Linearized model

Application: volt/var control

Closed-loop system is discrete-time dynamical system:

$$q_j(t+1) = \left[u_j \left(v_j(q(t)) - v_j^{\text{ref}} \right) \right]_{U_j}$$

where $v(q) = 2Xq + \tilde{v}$

<u>Assumptions</u>

- u_j are differentiable and $\left| u_j'(v_j) \right| \leq \alpha_j$
- u_j are strictly decreasing

Linearized model

Application: volt/var control

Theorem (Convergence)

If largest singular value $\sigma_{\max}(AX) < 1/2$ then

$A := \text{diag}\left(\alpha_j, j \in N\right)$

1. $\exists!$ equilibrium point $q^* \in U$
2. Closed-loop system converges to q^* geometrically, i.e.,

$$\|q(t) - q^*\| \quad \leq \quad \beta^t \, \|q(0) - q^*\| \quad \rightarrow \quad 0$$

for some $\beta \in (0,1)$

Linearized model

Application: volt/var control

Theorem (Optimality)

The unique equilibrium point $q^* \in U$ solves

$$\min_{q \in U} \quad \sum_j c_j(q_j) \quad + \quad q^T X q \quad + \quad q^T \Delta \tilde{v}$$

where $c_j(q_j) \; := -\int_0^{q_j} u_j^{-1}(\hat{q}_j)\, d\hat{q}_j$ and $\Delta \tilde{v} := \tilde{v} - v^{\text{ref}}$

Reverse engineering: by choosing a control function u_j, we implicitly choose a cost function $c_j\left(q_j\right)$ that the closed-loop equilibrium optimizes

Linearized model
Application: volt/var control

Theorem (Optimality)

The unique equilibrium point $q^* \in U$ solves

$$\min_{q \in U} \quad \sum_j c_j(q_j) \quad + \quad q^T X q \quad + \quad q^T \Delta\tilde{v}$$

where $c_j(q_j) \ := \ -\int_0^{q_j} u_j^{-1}(\hat{q}_j)\, d\hat{q}_j$ and $\Delta\tilde{v} := \tilde{v} - v^{\text{ref}}$

Forward engineering: Choose a cost function $c_j\left(q_j\right)$ and derive control functions u_j as distributed algorithm to solve the optimization problem

Summary

1. General network
   - Complex form, real form
2. Radial network
   - With and without shunt admittances
3. Equivalence
4. Backward forward sweep
5. Linearized model
   - Analytical solution, bounds, local volt/var control